Exercise 109

Evaluate $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$.

Solution

Evaluate the limit by rewriting the fraction.

$$\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} = \lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} \times \frac{\sqrt{1+\tan x}+\sqrt{1+\sin x}}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}$$

$$= \lim_{x\to 0} \frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to 0} \frac{\tan x-\sin x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to 0} \left(\frac{\tan x-\sin x}{x^3}\right)\left(\frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right)$$

$$= \left(\lim_{x\to 0} \frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\right)\left(\lim_{x\to 0} \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right)$$

$$= \left(\lim_{x\to 0} \frac{\frac{\sin x}{\cos x}-\sin x}{x^3} \times \frac{\cos x}{\cos x}\right)\left(\frac{1}{\sqrt{1+\tan 0}+\sqrt{1+\sin 0}}\right)$$

$$= \left(\lim_{x\to 0} \frac{\sin x-\sin x\cos x}{x^3\cos x}\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3\cos x}$$

$$= \frac{1}{2}\lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1-\cos x}{x^2\cos x}\right)$$

$$= \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1-\cos x}{x^2\cos x}\right)$$

$$= \frac{1}{2}(1)\left(\lim_{x\to 0} \frac{1-\cos x}{x^2\cos x} \times \frac{1+\cos x}{1+\cos x}\right)$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{1-\cos^2 x}{x^2\cos x(1+\cos x)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{\sin^2 x}{x^2\cos x(1+\cos x)}$$

Continue the simplification.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} &= \frac{1}{2} \lim_{x\to 0} \left(\frac{\sin^2 x}{x^2}\right) \left[\frac{1}{\cos x(1+\cos x)}\right] \\
&= \frac{1}{2} \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \left[\frac{1}{\cos x(1+\cos x)}\right] \\
&= \frac{1}{2} \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \left[\lim_{x\to 0} \frac{1}{\cos x(1+\cos x)}\right] \\
&= \frac{1}{2}(1)^2 \left[\frac{1}{(\cos 0)(1+\cos 0)}\right] \\
&= \frac{1}{2} \left[\frac{1}{(1)(2)}\right] \\
&= \frac{1}{4}
\end{aligned}$$